Excitation of Higher Order Modes in Spherical Cavities*

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Summary—**An analysis for determining approximately the optimum position of the exciting source inside a spherical cavity for exciting any TE or TM mode is presented. For any TE or TM mode the orientation of the exciting probe or loop is determined by maximizing the surface integral of $\overline{H}$ or line integral of $\overline{A}$ which is proportional to the excitation coefficient for the corresponding mode. Specific examples of mode discrimination by proper orientation of the exciting source are also included in the paper. Besides, graphs of the surface integral of $\overline{H}$ and the line integral of $\overline{A}$ for various modes are presented to indicate the variation of mutual inductance for any mode, for different positions of the exciting source.**

Higher order modes, that is, modes having resonant frequencies higher than that of the dominant mode, can be excited in a resonant cavity either by a magnetic loop or by a short antenna or even by an aperture coupling. In general, in order to excite a cavity for any particular mode by a magnetic loop the plane of the loop is placed normal to the magnetic lines for the corresponding mode and preferably at a region where the magnetic lines are dense. Similarly, in order to excite a cavity for any desired mode with a short antenna, the axis of the antenna is placed parallel to the electric lines for the corresponding mode. The problem of excitation of a cavity by a circular or linear slit or aperture can be considered as a dual problem of exciting the cavity by a loop or as a short antenna, where the excited electric and magnetic fields are interchanged.

Theoretically, an infinite number of modes can be excited by a magnetic loop and a short antenna. However, for a practical purpose it is often desirable to energize a cavity for a particular mode and to optimize the excitation for that mode by suitably orienting the exciting source.

The problem of orientation of the source to obtain optimum excitation for a desired higher order mode in case of rectangular and cylindrical cavities is rather straightforward. Often determining the location of the source becomes obvious from the field configuration of the desired mode. In a spherical cavity, however, the field configurations are a little more complex, and further insight into the field distribution becomes desirable. In the present paper, attempts have been made to present an analytical treatment of the excitation problem to indicate how the exciting source should be oriented in order to obtain an optimum excitation for any desired higher order mode in spherical cavities. Specific examples are shown to indicate the nature of discrimination between the desired and undesired modes when the desired mode is optimized.

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Excitation by a Magnetic Loop

Let a small wire loop carrying a uniform current be enclosed in a cavity to energize it. Following the method introduced by Condon[1] it is often convenient to assume that the total magnetic potential $\overline{A}$ can be expanded in terms of arbitrary orthogonal modes such that

$$\overline{A}(\overline{S}, t) = \sum_{k=0}^{\infty} \overline{A}_k(\overline{S}) T_k(t) \tag{1}$$

and

$$\iiint_V \overline{A}_n(\overline{S})\overline{A}_m(\overline{S})dV = 0 \quad \text{for} \quad n \neq m$$
$$= V \quad \text{for} \quad n = m \tag{2}$$

where $\overline{A}_K(\overline{S})$ is the resonant wave pattern of the magnetic potential for the kth mode and is a function of space alone, $T_k(t)$ is the corresponding time function, V is the volume of the enclosure, and k, n, m are integers. Similarly, one can also assume that the time and the space distribution of the exciting current can be expanded in terms of $I_k(t)$ and $\overline{A}_k(\overline{S})$, that is,

$$i(\overline{S}, t) = \sum_{k=0}^{\infty} I_k(t)\overline{A}_k(\overline{S}). \tag{3}$$

The coefficients $I_k(t)$ can now be evaluated by using the orthogonality property assumed in (2).

$$I_k(t) = \frac{1}{V}\iiint \overline{A}_k(\overline{S}) i(\bar{s}, t) dV. \tag{4}$$

It can be shown[1] that for a short magnetic loop in which the current is confined to a wire of small cross-sectional area

$$I_k(t) = \frac{I}{V} M_k \tag{5}$$

where M_k is the flux through the current carrying loop caused by unit amplitude of excitation of the kth mode. Therefore M_k is the mutual inductance between the exciting loop and the cavity for the kth mode.

The excitation coefficients for any mode can now be determined in terms of this mutual inductance, from the following differential equation derived from Maxwell's equation and (2) and (3).

$$\ddot{T}_k + \frac{\omega_k}{Q_k}\dot{T}_k + \frac{1}{\mu\epsilon}\left[\beta_k^2 T_k - I_k(t)\right] = 0 \tag{6}$$

where μ and ϵ are permeability and dielectric constants of free space respectively,

[1] E. U. Condon, "Forced oscillations in cavity resonators," *J. Appl. Phys.*, vol. 12, pp. 129–132; February, 1941.

$\omega_k = 2\pi \times$resonant frequency for the kth mode
$\beta_k = 2\pi/\lambda_k$
$\lambda_k =$ resonant wavelength for the kth mode
$Q_k =$ resonant Q of the cavity for the kth mode.

If a time variation of $e^{i\omega t}$ is assumed such that $T_k = T_k e^{i\omega t}$

$$-\omega^2 T_k + \frac{i\omega\omega_k}{Q_k} T_k + \omega_k^2 T_k = c^2 \frac{I}{V} M_k. \tag{7}$$

At resonance

$$|T_k|_{\omega\to\omega_k} = c^2 \frac{I}{V} \frac{M_k Q_k}{\omega_k^2}. \tag{8}$$

From (5)

$$M_k = \frac{1}{T_k} \iint_S \overline{H}_{kn} dS \tag{9}$$

where $\overline{H}_{kn}$ is the magnetic field intensity for the kth mode normal to differential area ds on the plane of the current loop.

For any particular cavity, Q_k, ω_k, etc. corresponding to the kth mode are usually fixed and hence

$$T_k = N \sqrt{\left| \iint \overline{H}_{kn} dS \right|} \tag{10}$$

N being a constant. Thus, the excitation for the kth mode can be optimized by making the surface integral in (10) maximum.

Excitation by a Probe

With the probe or electric coupling one can no longer assume that the resonator is charge free, and hence the introduction of both scalar and vector potential will be necessary to describe the entire electromagnetic field. Thus,

$$\overline{E} = -\mu\dot{\overline{A}} - \text{grad } \Phi \tag{11}$$

where Φ is the scalar potential function.

Substitution in Maxwell's equation gives

$$-\mu \frac{\partial}{\partial t} \text{div} \overline{A} - \nabla^2 \Phi = \rho/\epsilon$$

$$\nabla \times \nabla \times \overline{A} + \mu\epsilon \ddot{\overline{A}} = i(s, t) - \epsilon \text{ grad } \dot{\Phi}. \tag{12}$$

If $[i(\bar{s}, t) - \epsilon \text{ grad } \dot{\Phi}]$ is expanded such that

$$[i(\bar{s}, t) - \epsilon \text{ grad } \dot{\Phi}] = \sum_{k=0}^{\infty} I_k(t) \overline{A}_k(\overline{S})$$

the expression for $I_k(t)$ corresponding to (4) becomes

$$I_k = \frac{1}{V} \iiint_V [i(\bar{s}, t) - \epsilon \text{ grad } \dot{\Phi}] \overline{A}_k dV. \tag{13}$$

The contribution[1] due to the integral involving the displacement current vanishes, and so $I_k(t)$ remains same as that in (4).

$$I_k(t) = \frac{I}{V} \int \overline{A}_k \cdot dS \tag{14}$$

$$T_k = c^2 \frac{I}{V} \frac{\int \overline{A}_k \cdot dS}{\left[\omega_k^2 - \omega^2 - \frac{i\omega_k\omega}{Q_k}\right]}.$$

At resonance

$$|T_k| = c^2 \frac{I}{V} \frac{\left|\int \overline{A}_k \cdot dS\right|}{\omega_k^2} Q_k \tag{15}$$

which implies that T_k can be optimized by maximizing the integral.

Loop Coupling in Spherical Cavity

Let it be assumed that a spherical cavity is to be excited for a given TE mode by a loop. One may ask whether there is any optimum position at which the loop can be placed in order to obtain maximum excitation for the mode under consideration and if so how to determine this position.

For TE spherical modes $E_r = 0$, and the amplitude of the radial magnetic field is maximum near the axis $\phi = 0$. Hence, it will be desirable to place the plane of the loop perpendicular to this axis for the corresponding mode. Let the coordinates of the center of the loop as shown in Fig. 1 be $(\alpha, 0, 0,)$, and let a and δ be the radii of the sphere and the loop respectively.

The magnetic field equations[2] corresponding to any TE n, m, o mode are

$$H_r = i(n+1)nk_n \frac{\beta}{\omega\mu} \left[\frac{j_n(\beta r)}{\beta r}\right] P_n(\cos\theta) e^{i\omega t}$$

$$H_\theta = \frac{i\beta k_n}{\omega\mu} P_n'(\cos\theta) \frac{1}{\beta r} \frac{\partial}{\partial(\beta r)} [\beta r j_n(\beta r)] e^{i\omega t} \tag{16}$$

$$H_\phi = 0.$$

where k_n is proportional to the nth-mode amplitude.

From (14), it appears that the excitation for nth mode can be optimized by maximizing

$$\iint_S \overline{H}_{nn} dS.$$

But as symmetry in the ϕ direction is assumed

$$\iint_S \overline{H}_{nn} dS = 2\pi \int_0^\delta \overline{H}_{nn} \rho d\rho$$

where $\overline{H}_{nn}$ is the magnetic intensity for the nth mode normal to the loop. From Fig. 1, $\rho = \alpha \tan\theta$; $\gamma' = \alpha/\cos\theta$

[2] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1941. The units are in unrationalized mks system which should not make any difference for the purpose of this analysis.

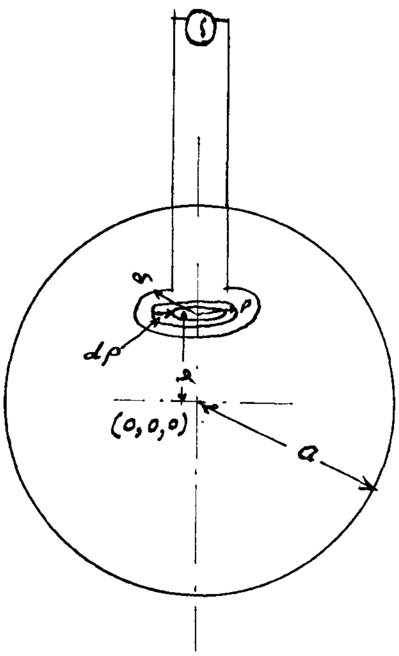

Fig. 1—Excitation of a spherical cavity by a loop.

$$\iint_S \overline{H}_{nn} dS = i(n^2+n)k_n \frac{\beta}{\omega\mu}\frac{2\pi\alpha}{\beta} \cdot \int_0^{\theta_0} j_n\left(\frac{\beta\alpha}{\cos\theta}\right) P_n(\cos\theta)\frac{\sin\theta}{\cos^2\theta}d\theta \quad (17)$$

where

$$\theta_0 = \tan^{-1}\delta/\alpha.$$

Expanding P_n (cos θ) and interchanging the order of integration and summation one obtains

$$\iint \overline{H}_{nn} dS = i2k_n \frac{\pi\alpha}{\omega\mu}\sum_{p=0}^{t}\frac{1}{\Gamma(1/2)}\frac{(-1)^p\Gamma(n-p+\frac{1}{2})2^{n-2p}}{\Gamma(n-2p+1)p!}(n^2+n) \cdot \int_0^{\theta_0} j_n\left(\frac{\beta\alpha}{\cos\theta}\right)(\cos\theta)^{n-2p-2}\sin\theta d\theta \quad (18)$$

where

$$t = \frac{n}{2} \quad \text{for } n \text{ even}$$

$$= \frac{n-1}{2} \quad \text{for } n \text{ odd.}$$

But

$$j_n\left(\frac{\beta\alpha}{\cos\theta}\right) = \sqrt{\frac{\pi\cos\theta}{2\beta\alpha}}J_{n+1/2}\left(\frac{\beta\alpha}{\cos\theta}\right) = \sum_{q=0}^{\infty}\frac{\sqrt{\pi}(-1)^q\left(\frac{\beta\alpha}{2}\right)^{n+2q}}{2\cdot q!\Gamma(q+n+3/2)}(\cos\theta)^{-n-2q}.$$

Substituting this expression for $j_n(\beta\alpha/\cos\theta)$ in (18) and evaluating the integral one obtains

$$\left|\iint_S \overline{H}_{nn} dS\right| = k_n(n^2+n)\frac{\pi\alpha}{\omega\mu}\sum_{p=0}^{t}\frac{1}{\Gamma(1/2)}\frac{(-1)^p\Gamma(n-p+1/2)2^{n-2p}}{p!\Gamma(n-2p+1)} \cdot \sum_{q=0}^{\infty}\frac{\sqrt{\pi}}{2q!}\frac{(-1)^q\left(\frac{\beta\alpha}{2}\right)^{n+2q}}{\Gamma(q+n+3/2)} \cdot \left[\frac{1}{q+p+1/2}\left\{\left(1+\frac{\delta^2}{\alpha^2}\right)^{q+p+1/2}-1\right\}\right]. \quad (19)$$

If

$$\delta \ll \alpha, \quad \left|\iint_S \overline{H}_{nn} dS\right| = k_n\frac{A\beta}{\omega\mu}(n^2+n)\frac{j_n(\beta\alpha)}{\beta\alpha} \quad (20)$$

where A is the area of the loop.

The condition for maximum excitation[3] is obtained by solving the following equation for α

$$\frac{\delta}{\delta(\beta\alpha)}\left[\iint_S \overline{H}_{nn} dS\right] = 0 \quad (21)$$

which yields the transcendental equation

$$\alpha\beta = n\frac{j_n(\beta\alpha)}{j_{n+1}(\beta\alpha)} \quad (22)$$

It should be noted that (22) shows the optimum position for the loop to obtain maximum excitation for the nth mode, for $m=1$, provided $j_n(\beta\alpha)\neq 0$ when α is non zero. This exception is seen to be obvious from (20) since the excitation approaches zero at the "zeros" of $j_n(\beta\alpha)$. Because of symmetry it will be desirable to place similar loop or loops at the corresponding positions where $m>1$.

Let the case of exciting the dominant mode be considered as an example. From (16) one finds that the first resonance occurs at

$$\tan\beta a = \beta a$$

$$\beta a = 4.493.$$

The condition for the maximum excitation of the dominant mode is obtained from (22), which, for $TE_{1,1,0}$ mode,[4] yields $\alpha\beta=0$.

In order to observe the nature of discrimination against undesired modes when the desired mode is optimized, the relative variation of $|\iint\overline{H}_{nn}dS|$ for various positions of the exciting source and for different modes are shown in Fig. 2. It appears that if it is desired to excite the dominant mode in a spherical cavity,

[3] If δ is not too small in comparison with α the condition of maximum excitation is obtained by differentiating (19) directly with respect to α and setting the result equal to zero.

[4] From (20) it may appear that this solution is not permissible as $\delta\ll\alpha$ and $\alpha=0$, also $\delta>0$. But from the original equation for the magnetic field intensity as shown in (16) it can be shown that this solution is valid.

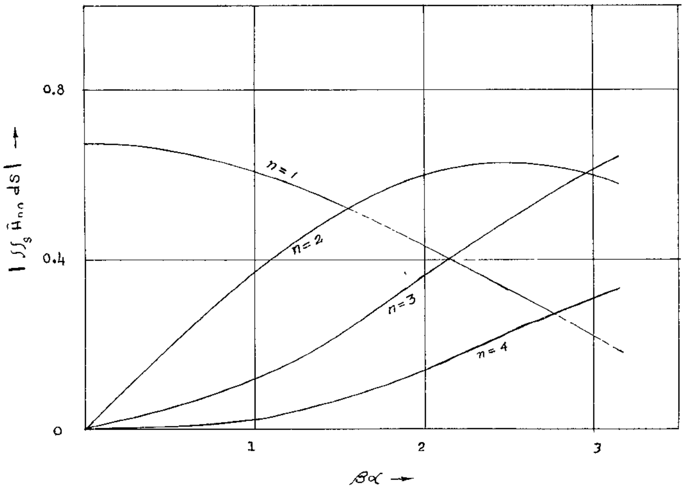

Fig. 2—Relative variation of $\int\int H_{nn} dS$ for various positions of the loop and for different TE modes.

the best position at which the exciting loop can be placed is at the center of the sphere. An analysis for the excitation of the dominant mode in a spherical cavity by a loop placed at the center of the sphere has been made by Schelkunoff.[5]

Similarly when it is desired to optimize TE $_{2,1,0}$ mode, the transcendental (22) yields a numerical solution

$$\beta a = 2.5.$$

It can be seen that considerable discriminations can be made against the undesired modes when $\beta\alpha$ is chosen equal to 2.5 for $TE_{2,1,0}$ mode. The optimum positions of the loop for other modes can be determined similarly. The surface integral $\int\int \overline{H}_{nn} dS$ when divided by T_n, as shown in (9) yields the mutual inductance for the nth mode. Thus Fig. 2 may be helpful to evaluate the mutual inductance of any mode for different positions of the loop at any frequency.

It may be remarked here that in the preceding analysis, the effect of distortion of the field due to the loop has been neglected. Furthermore, the loop has to be supported in the desired position inside the cavity by some means which may cause distortion. However, if δ is made reasonably small as is assumed in the analysis and the loop is supported by means of two parallel wires spaced very close to each other, the wave patterns will not change appreciably. If the total distortionless field components say H_r due to all the modes is represented as

$$H_r = \sum a_n \frac{(n^2 + n)}{\beta^2 r^2} [(\beta r) j_n(\beta r)] P_n(\cos\theta)$$

the distorted field under the above said conditions may be represented as

$$\overline{H}_r = \sum \bar{a}_n \frac{(n^2 + n)}{\beta^2 r^2} [(\beta r) j_n(\beta r)] P_n(\cos\theta)$$

[5] S. A. Schelkunoff, "Electromagnetic Waves," D. Van Nostrand Co., Inc., New York, N. Y., p. 298; January, 1956.

where

$$a_n \neq \bar{a}_n.$$

This inequality, however, will not affect the analysis for the optimization of the coupling for any mode considered independently.

Excitation of Higher Order Transverse Magnetic Modes

For transverse magnetic modes $H_r = 0$, and the loop excitation as described in the preceding section can no longer be used to excite higher order TM modes in the spherical cavity. A short antenna or probe (Fig. 3), introduced through the periphery of the sphere radially at $\phi = 0$, can be used, however, to excite spherical TM modes.

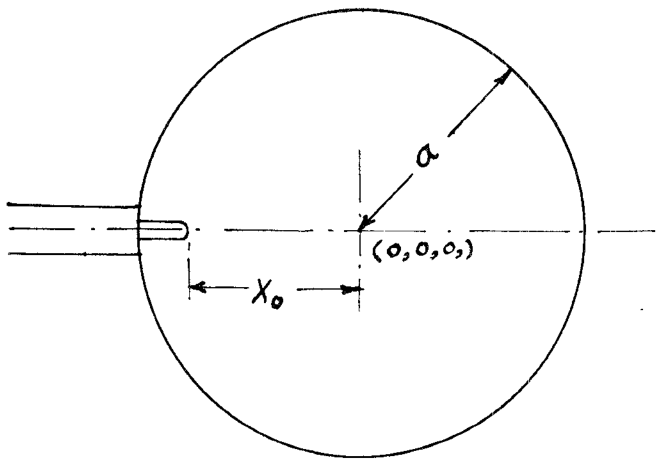

Fig. 3—Excitation of a spherical cavity by a probe.

For probe excitation as discussed in the previous section one finds

$$\overline{E} = -\mu\dot{\overline{A}} - \text{grad}\ \Phi.$$

When a time variation of $e^{i\omega t}$ is assumed

$$\overline{A} = \frac{i}{\omega\mu} [\overline{E} + \text{grad}\ \Phi]. \quad (23)$$

If the depth of penetration is the only variable to cause any change in the excitation T_k, $\int \overline{A}_k\ dS$ becomes maximum when

$$\int_{\beta X_0}^{\beta a} \frac{j_n(\beta r)}{\beta r} d(\beta r) = k \int \overline{A}_n \cdot dr \quad (24)$$

is maximum, K being a constant.

As the integral exists everywhere in the interval $[\beta a, \beta X_0]$, the integral can be expressed as

$$I = F(\beta a) - F(\beta X_0) \quad (25)$$

where

$$F(x) = \left[\int \frac{j_n(\xi)}{\xi} d\xi \right]_{\xi = x}.$$

In order to determine optimum depth of penetration which will optimize the line integral of $\overline{A}_n$, one can set

$$\frac{dI}{dX_0} = 0 \tag{26}$$

and obtain a solution for X_0 which will yield a maximum value of the integral I. From (25) and (26) it will appear that $\int \overline{A}n \cdot dS$ is maximum when

$$j_n(\beta X_0) = 0. \tag{27}$$

Fig. 4 shows the relative variation of $\int A_k \cdot dS$ for $k=1, 2$, which correspond to $TM_{1,1,0}$, and $TM_{2,1,0}$ modes, when the cavity is designed for TM_{110} resonance which occurs at

$$\beta a = 2 \cdot 74.$$

It may be remarked that Fig. 4 not only indicates the nature of discrimination against undesired $TM_{2,1,0}$ mode, but also shows the variation of the mutual inductance for $TM_{1,1,0}$ and $TM_{2,1,0}$ modes for different probe lengths.

It is obvious that for any mode, $X_0 = 0$, which corresponds to a probe length equal to the radius of the sphere, is a solution of (26). In fact, for any mode such as the dominant mode where no zero of $j_n(\beta X_0)$ exists between βa and 0, the probe has to be extended up to the center of the sphere in order to obtain the maximum excitation. However, the probe length can be reduced for any higher order mode ($n>1$) if the root of (27) be such that $\beta a > \beta X_0 > 0$. Under these circumstances it will not only be desirable but also necessary to extend the probe up to $X_0 \neq 0$, in order to avoid simultaneous excitation of the undesired modes.

It should be noted that the distortion of the wave pattern of the field components has not been considered in the present analysis. Thus the solution of the problem so obtained is an approximate one, and it approaches the true solution when the probe is made infinitely thin. The field distortion due to relative change in amplitude of the different mode components, however, will not affect the analysis for the optimization of the excitation coefficient as stated before.

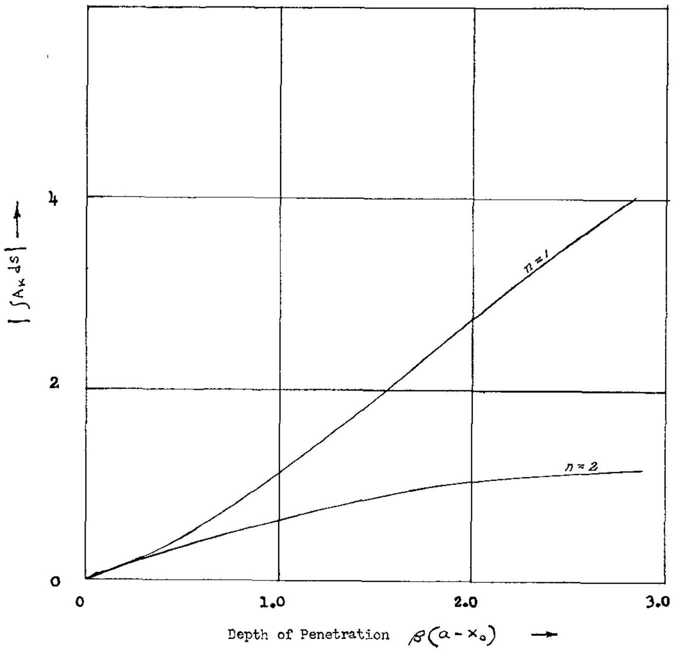

Fig. 4—Relative variation of $\int A_K dS$ for different probe length when the cavity is designed for TM_{110} mode.

Conclusion

A theoretical analysis for the excitation of higher order TE and TM modes in a spherical cavity is presented to indicate the optimum orientation of the loop or probe so as to obtain maximum excitation for any desired mode. A general expression for the position of a magnetic loop which can be used to excite any particular radial TE mode in a spherical cavity is found so that the excitation of the cavity for this particular mode becomes maximum at the corresponding position. Similarly, an expression for the probe length, in case of an antenna coupling, which will yield optimum excitation for any desired TM mode, is obtained. Although the problem of excitation of higher order modes in the spherical cavity alone has been discussed in the present paper, the basic principles can easily be extended to the problems of optimum excitation in rectangular, cylindrical, and elliptical cavities.